# BOSONIC FIELD PROPAGATORS ON ALGEBRAIC CURVES

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ABSTRACT

In this paper we investigate massless scalar field theory on non-degenerate algebraic curves. The propagator is written in terms of the parameters appearing in the polynomial defining the curve. This provides an alternative to the language of theta functions. The main result is a derivation of the third kind differential normalized in such a way that its periods around the homology cycles are purely imaginary. All the physical correlation functions of the scalar fields can be expressed in terms of this object. This paper contains a detailed analysis of the techniques necessary to study field theories on algebraic curves. A simple expression of the scalar field propagator is found in a particular case in which the algebraic curves have $Z_n$ internal symmetry and one of the fields is located at a branch point.

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## 1. INTRODUCTION

In the last decade there has been a growing interest in the applications of algebraic curves to several problems in theoretical physics, ranging from the theory of strings to condensed matter physics [1]–[11]. Algebraic curves, defined as the region in which a complex polynomial of two variables vanishes, provide an explicit and useful representations of Riemann surfaces. Thus, one can take advantage of well established results and theorems of algebraic geometry [12]–[14]. So far, mainly applications involving curves with cyclic monodromy group have been studied neglecting other, more interesting examples of algebraic curves. The reason is that computations on general algebraic curves are intrinsically complicated. For instance, it is not known how to express the theta functions [12] in terms of the parameters appearing in the polynomial which define the curve. This is a great handicap, since theta functions are the main building blocks in the construction of tensors with given zeros and poles on Riemann surfaces. Other problems are connected with the analytical continuation of multivalued functions on the complex plane. For example, any attempt of setting up an operator formalism on algebraic curves based on the monodromy of the fields [4],[15], leads to differential equations of the Riemann monodromy problem type which are too difficult to be solved in practice. On the contrary, there exist several ways to expand free conformal fields on a Riemann surface using the formalism of theta functions [16]–[24].

Some progress in the physical applications of general algebraic curves can be obtained by studying simple conformal field theories. For example, the correlation functions of the $b-c$ systems have been explicitly computed by means of an operator formalism in [25]–[26]. The construction of a generalized Laurent basis in order to expand meromorphic tensors has been achieved in [26]. In the particular case of nonabelian monodromy groups $D_n$, $n = 2, 3, 4, \ldots$, it is possible to show that these elements are connected to a Riemann monodromy problem [27]. The relations between free $b-c$ systems on algebraic curves and new conformal field theories on the complex plane has been explored in [28],[29]. The operator formalism of [26] has been extended to the $\beta-\gamma$ systems [30],[31] with integer spins in ref. [32].

In this paper we study correlation functions of free massless scalar field theory on algebraic curves. The form of the curve is restricted only by a requirement of non-degeneracy. The problem is reduced to the computation of a differential of the third kind normalized so that its periods along the homology cycles are purely imaginary [13]. To this purpose, we construct here a differential of the third kind using the techniques developed in ref. [26]. Its periods along the homology cycles are then fixed by adding suitable linear combinations of holomorphic differentials. To pick up exactly the linear combination which makes all the periods imaginary is however not a simple task. As a matter of fact, on general algebraic curves it is not even known how to construct a basis of homology cycles. To overcome this difficulty, we exploit here a set of Riemann bilinear identities [12], the derivation of which is presented for a sake of completeness in Appendix A. These relations uniquely determine the desired differential of the third kind and provide an explicit expression for it in terms of the parameters which characterize the curve[3].

The material contained in this paper is organized as follows. In Section 2 we develop the necessary techniques to study field theories on algebraic curves. With respect to ref. [26], the curves are entirely general apart from the non-degeneracy. To simplify our analysis, the defining polynomials are reduced to a canonical form introducing a set of suitable transformations. The relevant differentials like the Weierstrass kernel and the holomorphic differentials are derived together with their divisors. The genus of the curve is obtained from the number of holomorphic differentials. This result is checked using a Baker's method [34]. In Section 3 the theory of massless scalar field theories is briefly reviewed following [35]. It is shown that all the correlation functions can be written in terms of the canonical differential of the third kind with purely imaginary periods around the non-trivial monodromy cycles. This canonical differential is constructed as a a linear combination of Weierstrass kernels and holomorphic differentials. The coefficients of the linear combination are determined by means of a system of Riemann bilinear equations, which are satisfied if and only if the differential of the third kind has purely imaginary periods. The correlation functions of the scalar fields obtained in this way explicitly depend on the parameters of the polynomial which defines the algebraic curve as desired. Some examples are worked out in Section 4. The latter Section contains also the derivation of the massless scalar field propagator on $Z_n$ symmetric algebraic curves [4] supposing that one of the scalar fields is located at a branch point. This particular case is of some relevance in string theories as explained in ref. [8]. The Riemann bilinear relations used in Section 3

[3] Let us notice that similar methods, using integral identities like the Riemann bilinear relations, have been already used in the computation of scalar Green functions on hyperelliptic curves [33].

are derived in Appendix A, while in Appendix B it is shown how integrals on an algebraic curve can be rewritten as integrals of multivalued volume forms over the complex sphere following [14]. Finally, we discuss our results and their possible applications.

## 2. THE LANGUAGE OF ALGEBRAIC CURVES

Let $\mathbf{C}$ be the complex plane and $\mathbf{CP}^1$ the complex projective line which coincides with the compactified complex plane $\mathbf{C}\cup\{\infty\}$. We will describe Riemann surfaces as $n-$sheeted branched covers of $\mathbf{CP}^1$ [14]. The latter are given as algebraic curves defined as the locus of points $(z,y)\in\mathbf{CP}^1\times\mathbf{CP}^1$ for which the following equation is satisfied:

$$F(z,y)=0 \tag{2.1}$$

Here $F(z,y)$ denotes a polynomial of the form:

$$F(z,y)=P_n(z)y^n+P_{n-1}(z)y^{n-1}+\ldots+P_1(z)y+P_0(z)=0 \tag{2.2}$$

with $P_s(z)=\sum\limits_{m=0}^{n_s}A_{s,m}z^m$ for $s=0,\ldots,n$ and $n_s\in\mathbf{N}$. By a well known theorem, apart from subtleties coming from singular points, any Riemann surface can be expressed as an algebraic curve of this kind. Thus, from now on, we will use the words Riemann surface, $n-$sheeted branched covers of $\mathbf{CP}^1$ and algebraic curves interchangeably. The best known algebraic curves are the hyperelliptic curves, whose polynomials $F(z,y)$ are simply given by: $F(z,y)=y^2-P_0(z)$. Also the slightly more general $Z_n$ symmetric curves will be often mentioned here. They are characterized by $F(z,y)=y^n-P_0(z)$.

One can solve eq. (2.1) with respect to $y$ and obtain in this way a function $y(z)$ which is multivalued on $\mathbf{CP}^1$. Its $n$ branches are interchanged at a set of $N_{bp}$ branch points $a_i,\ldots,a_{N_{np}}\in\mathbf{CP}^1$ as will be discussed below. In the following, the branches of $y(z)$ will be denoted with the symbol $y^{(\alpha)}(z)$, $\alpha=0,\ldots,n-1$ and the first letters of the Greek alphabet will be used as branch indices. To simplify our analysis, we assume that $P_n(z)=1$ in eq. (2.2). There is no loss of generality in this assumption. As a matter of fact, if $P_n(z)\neq 1$, one can always perform the change of variables

$$\tilde{y}(z)=y(z)P_n(z) \tag{2.3}$$

which does not affect the monodromy properties of $y(z)$, so that both $\tilde{y}(z)$ and $y(z)$ are meromorphic functions on the same Riemann surface. However, it is easy to realize that

$\tilde{y}$ satisfies the equation $\sum_{i=0}^{n}\tilde{P}_{n-i}(z)\tilde{y}^{n-i} = 0$, where now $\tilde{P}_n(z) = 1$ as desired and $\tilde{P}_{n-i}(z) = P_{n-i}(z)P_n^{i-1}(z)$ for $i = 1,\ldots,n$.

Further, we require that none of the branch points is in $z = \infty$. Again, this condition does not restrict the generality of our discussion as we will show below. First of all, the presence of a branch point at infinity can be detected investigating the fact that, for large values of $z$, the function $y(z)$ exhibits the following behavior:

$$y(z) \underset{z\to\infty}{\sim} cz^p + \text{lower order terms} \tag{2.4}$$

Substituting eq. (2.4) in eq. (2.1) and solving the latter at the leading order in $z$, one determines the allowed values of the constants $c$ and $p$. A branch point at $z = \infty$ is indicated by noninteger solutions for $p$. If this is the case, it is always possible to perform a $SL(2,\mathbf{C})$ transformation in $z$, in such a way that the branch point at infinity is moved to a finite region of the plane. Of course, the condition $P_n(z) = 1$ does no longer hold in the new variable, but it can be easily restored with the aid of the transformations (2.3) in $y$. Since the letter transformation does not affect the monodromy properties of $y$, the branch point at $z = \infty$ cannot be reintroduced.

We are now ready to study the finite branch points $a_1,\ldots,a_{N_{bp}}$. Supposing that the curve (2.1) is nondegenerate[4], they are the solutions of of the following system of equations:

$$F(z,y) = F_y(z,y) = 0 \tag{2.5}$$

where $F_y(z,y) = dF(z,y)/dy$. It is useful to eliminate from eq. (2.5) the variable $y$. As an upshot, one obtains a polynomial equation in $z$ of the kind $r(z) = 0$. Apart from very special curves, in which $r(z)$ has multiple roots, its degree coincides with the number of branch points $N_{bp}$. Let us notice that it is possible to derive the resultant $r(z)$ of eqs. (2.5) explicitly using the dialitic method of Sylvester [36].

To each branch point $a_i$ one can associate an integer $\nu_i$, called the ramification index and defined as the number of branches of $y(z)$ that are exchanged at that branch point. Clearly, $2 \leq \nu_i \leq n$. At a branch point of ramification index $\nu_i$, the polynomial $F(z,y)$ vanishes together with its first $\nu_i - 1$ partial derivatives in $y$. The genus $g$ of the Riemann surface (2.1) is related to the ramification indices of the branch points and the number of sheets $n$ composing the curve as follows (Riemann–Hurwitz theorem):

$$2g - 2 = -2n + \sum_{s=1}^{L}(\nu_s - 1) \tag{2.6}$$

[4] See for instance ref. [14] for the definition of nondegeneracy.

The genus $g$ can be explicitly computed once the form of the Weierstrass polynomial is known exploiting the Baker's method, see ref. [34]. This will be done at the end of this Section.

On a Riemann surface $\Sigma_g$ represented as an $n-$sheeted cover of $\mathbf{CP}^1$ there is a "canonical" complex structure inherited from $\mathbf{CP}^1$. A possible atlas on $\Sigma_g$ is the following. Let us put $R = \max|a_i|$ and $\rho = \min|a_i - a_j|$ for $i, j = 1, \ldots, N_{bp}$. Near a branch point $a_i$ of ramification index $\nu_i$, or more precisely in the open disk $|z - a_i| < \rho$, we choose the local coordinate $\xi^{\nu_i} = z - a_i$. For $|z| > R$, the local coordinate is $z' = 1/z$. Let us notice that on the algebraic curve the set $|z| > R$ corresponds to an union of $n$ disjoint discs. On the remaining open sets the local coordinate is $z$ (the same letter $z$ denotes here coordinates on $n$ different branches of $\Sigma_g$; this convention is very useful and, hopefully, does not generate ambiguities).

In the rest of this Section, we discuss the construction of the relevant meromorphic tensors and the computation of their divisors. We are interested in tensors of the kind $T^{(\alpha)}(z)dz^\lambda$, with $\lambda$ upper or lower indices depending on the sign of $\lambda = 0, \pm 1, \pm 2, \ldots$. The treatment of tensors characterized by half-integer values of $\lambda$ is possible only on hyperelliptic curves and will not be considered here. The meromorphic functions correspond to the case $\lambda = 0$. The branch index $\alpha$ has been added to recall that a tensor $T$ is multivalued on $\mathbf{CP}^1$ due to its dependence on $y(z)$. To any meromorphic tensor $Tdz^\lambda$ with zeros at $z_r$ of order $k_r$ and poles at $p_s$ of order $l_s$, one can associate a divisor $[T]$ [12] :

$$[T] = \sum_r k_r z_r - \sum_s l_s p_s. \tag{2.7}$$

$k_r$ and $l_s$ are integers while $z_r$ and $p_s$ denote points on the algebraic curve The degree of the divisor $[T]$ is defined as follows:

$$deg[T] = \sum_r k_r - \sum_s l_s. \tag{2.8}$$

The most general tensor on an algebraic curve is of the form:

$$T^{(\alpha)}(z)dz^\lambda = Q(z, y^{(\alpha)}(z)) \frac{dz^\lambda}{[F_y(z, y^{(\alpha)}(z))]^\lambda} \tag{2.9}$$

where $Q(z, y)$ is a rational function of $z$ and $y$. The reason for which the factor $[F_y(z, y^{(\alpha)}(z))]^{-\lambda}$ has been singled out in (2.9) will soon become clear. From eq. (2.9) it is evident that, in order to construct tensors on an algebraic curve with poles and zeros at given points, it is necessary to know the divisors of the basic building blocks $dz$, $y$ and $F(z,y)$. This can be done quite explicitly for the general polynomials described by eq. (2.2) if $P_n(z) = 1$ and if there are no branch points at infinity. We only need the additional assumption that the polynomials $P_1(z)$ and $P_0(z)$ appearing in $F(z,y)$ have no roots in common. In this way, eq. (2.1) is approximated for small values of $y$ by the relation $y \sim -P_0(z)/P_1(z)$. Therefore, the zeros $q_1,\ldots,q_{n_0}$ of $y(z)$ occur for values of $z$ corresponding to the roots of $P_0(z)$. To study the behavior of $y(z)$ at infinity we try the ansatz (2.4) in eq. (2.1). If we retain only the leading order terms of $y(z)$ and of the polynomials $P_s(z)$ appearing in (2.2), then eq. (2.1) is approximated by:

$$c^n z^{pn} + \ldots + A_{s,n_s} c^s z^{ps+n_s} + \ldots + A_{0,n_0} z^{n_0} = 0 \tag{2.10}$$

Since by assumption $y(z)$ is not branched at infinity, there should be $n$ different solutions for $c$ that satisfy (2.10). Clearly, this can be true only if the first and last monomials $c^n z^{pn}$ and $A_{0,n_0} z^{n_0}$ entering in eq. (2.10) are growing with the same power of $z$ near the point $z = \infty$, i. e. $z^{pn} \sim z^{n_0}$. Moreover, all the other leading order monomials appearing in eq. (2.10) must not contain higher order powers in $z$, i. e.:

$$ps + n_s \leq n_0 \qquad\qquad s = 1,\ldots,n-1 \tag{2.11}$$

Thus, we obtain for $p$ the following result:

$$p = \frac{n_0}{n} = 1, 2, \ldots \tag{2.12}$$

This implies that the integer $n_0$ is a multiple of $n$.

In this way we have derived the divisor of $y$:

$$[y] = \sum_{r=1}^{n_0} q_r \; - \; \sum_{j=0}^{n-1} \frac{n_0}{n} \infty_j \tag{2.13}$$

The symbols $\infty_j$ denote the points on the curve corresponding to the point $z = \infty$ in $\mathbf{CP}_1$. In the covering of the algebraic curve described above they belong to the $n$ disjoint discs where the condition $|z| > R$ is satisfied. As we see from the above equation, the degree of the divisor $[y]$ is zero as expected for a meromorphic function. Analogously, it is possible to compute the divisors of $F_y(z,y)$ and $dz$:

$$[F_y] = \sum_{r=1}^{n_{bp}} (\nu_r - 1) a_r \; - \; (n-1) \sum_{j=0}^{n-1} \frac{n_0}{n} \infty_j \tag{2.14}$$

$$[dz] = \sum_{r=1}^{n_{bp}} (\nu_r - 1) a_r \; - \; 2 \sum_{j=0}^{n-1} \infty_j \tag{2.15}$$

The details are explained in ref. [26]. Exploiting the above divisors, one is able to prove that, if $\lambda \geq 2$, the following tensor has only a simple pole at the point $z = w$ on the sheet $\alpha = \beta$:

$$K_\lambda^{(\alpha\beta)}(z, w) dz^\lambda = \frac{1}{z - w} \frac{F(w, y^{(\alpha)}(z))}{[y^{(\alpha)}(z) - y^{(\beta)}(w)]} \frac{dz^\lambda}{[F_y(z, y^{(\alpha)}(z))]^\lambda} \tag{2.16}$$

where the indices $\alpha$ nd $\beta$ label the branches in $z$ and $w$ respectively. The tensor $K_\lambda^{(\alpha\beta)}(z, w) dz^\lambda$ will be hereafter called the Weierstrass kernel. If $\lambda = 1$, it is easy to check that

$$\nu_{uw}^{(\alpha\beta\gamma)}(z) dz = K_1^{(\alpha\beta)}(z, u) dz - K_1^{(\alpha\gamma)}(z, w) dz \tag{2.17}$$

is a differential of the third kind [5] with two simple poles in $z = u$ and $z = w$ on the sheets $\alpha = \beta$ and $\alpha = \gamma$ respectively. The residue of $\nu_{uw}^{(\alpha\beta\gamma)}(z) dz$ is $+1$ at $z = u$ and $-1$ at $z = w$. For our purposes we will also need differentials of the first kind, or holomorphic one forms and differentials of the second kind, which consists of meromorphic one forms with vanishing residue. Any meromorphic differential can be decomposed in terms of the elements of a generalized Laurent basis given by [25]:

$$f_{k,i}(z) dz = \frac{z^{-i-1} y^{n-1-k}(z)}{F_y(z, y(z))} dz \qquad \begin{cases} i = 0, \pm 1, \pm 2, \ldots \\ k = 0, \ldots, n-1 \end{cases} \tag{2.18}$$

For the functions, instead, it is possible to use the following basis:

$$\phi_{k,i}(z) = z^{-i} \left( y^k(z) + y^{k-1}(z) P_{n-1}(z) + y^{k-2}(z) P_{n-2}(z) + \ldots + P_{n-k}(z) \right) \tag{2.19}$$

where $k$ and $i$ take the same values as in eq. (2.18). The elements of the basis (2.19) look apparently complicated, yet they are very convenient in order to expand the differentials of the third kind (2.17). As a matter of fact, it is possible to show that [26]:

$$\nu_{uw}^{(\alpha\beta\gamma)}(z) dz = \sum_{k=0}^{n-1} \frac{f_{k,-1}^{(\alpha)}(z) \phi_{k,0}^{(\beta)}(u)}{z - u} dz - \sum_{k=0}^{n-1} \frac{f_{k,-1}^{(\alpha)}(z) \phi_{k,0}^{(\gamma)}(w)}{z - w} dz \tag{2.20}$$

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[5] Following the classification of [37], here third kind differentials are defined as meromorphic differentials with at most simple poles. This definition differs for instance from that of [12].

To construct a basis of holomorphic differentials it is sufficient to find all possible values of $s$ and $k$ in eq. (2.18) for which $f_{k,s}(z)dz$ has no poles. It is easy to check with the help of eqs. (2.13)–(2.15) that such a basis is given by:

$$\Omega_{(j,s_j)}(z)dz = \frac{z^{s_j} y^j(z)}{F_y(z,y(z))} dz \tag{2.21}$$

where, for $p = 1, 2, \ldots$

$$j = 0, \ldots, n-2-\delta_{p,1} \qquad s_j = 0, \ldots, (n-1-j)p-2 \qquad \delta_{p,1} = \begin{cases} 1 & \text{if } p = 1 \\ 0 & \text{if } p > 1 \end{cases} \tag{2.22}$$

The number of the above holomorphic differentials coincides with the genus of the curve (2.2). A straightforward computation gives:

$$g = \frac{pn(n-1) - 2(n-1)}{2} \tag{2.23}$$

The above calculation of $g$ can be verified using the already mentioned method of Baker. To this purpose, let us consider a two dimensional integer lattice on the $x - y$ plane. One draws a triangle $OAC$ with vertices at the points $O = (0,0)$, $A = (a,0)$ and $C = (0,b)$, where $a = n$ and $b = pn$. According to Baker's method, the genus of the curve (2.2) with the additional constraints (2.11)–(2.12) and $P_n(z) = 1$ coincides with the number $\tilde{g}$ of lattice points contained inside the area of this triangle, the boundary excluded. Let us evaluate $\tilde{g}$. First of all, one counts the number of lattice points $T$ inside the rectangle $OABC$, where $B = (a,b)$, excluding also the boundary. Then one subtracts the number of links $D$ lying on the diagonal $AC$. One finds that $T = (pn-1)(n-1)$ and $D = n-1$. Clearly

$$\tilde{g} = \frac{T - D}{2} \tag{2.24}$$

and it is easy to see that this gives for the genus of the curve (2.2) exactly the result of eq. (2.23), i. e. $\tilde{g} = g$.

Finally, a possible metric on the curve (2.2) is given by

$$ds^2 = \tilde{\rho}_{z\bar{z}} dz d\bar{z} = (1 + z\bar{z})^{\mu} \frac{dz d\bar{z}}{|F_y(z,y)|^2} \qquad \mu = p(n-1) - 2 \tag{2.25}$$

## 3. SCALAR GREEN FUNCTIONS ON ALGEBRAIC CURVES

In this Section we consider the free bosonic scalar field theory described by the action:

$$S = \int_{\Sigma_g} d^2 z \partial\varphi \bar{\partial}\varphi \tag{3.1}$$

defined on a general algebraic curves $\Sigma_g$ discussed above. A conformal metric on $\Sigma_g$ is understood in (3.1) . Moreover the $d^2z$ is a shorthand notation for $\frac{1}{2i}dzd\bar{z}$.

A scalar field in the two dimensional field theory (3.1) satisfies the Poisson equation can can be interpreted as an electrostatic potential of a Coulomb system of charges. $\partial_z\varphi dz$ is a meromorhic differential with the sum of residua at poles equal zero. Consequently the sum of charges on the Riemann surface must be zero as well. Thus we consider a system of charges $q_i$ set in the positions $z_i$, $i = 1,\dots,M$ interacting with another system of charges $q'_j$ located at the points $w_j$, $j = 1,\dots,N$ and satisfying the relations $\sum\limits_i q_i = \sum\limits_j q'_j = 0$. The corresponding correlation function:

$$G(z_1\dots z_M; w_i\dots w_N) = \sum_{i=1}^{M}\sum_{j=1}^{N} q_i\langle\varphi(z_i,\bar{z}_i)\varphi(w_j,\bar{w}_j)\rangle q'_j \tag{3.2}$$

can be computed once the following Green function is known:

$$G_z(z; u, w) = \langle\partial_z\varphi(z,\bar{z})\left[\varphi(u,\bar{u}) - \varphi(w,\bar{w})\right]\rangle \tag{3.3}$$

As a matter of fact, one can prove the following relation:

$$G(z_1\dots z_M; w_i\dots w_N) = \sum_{i=1}^{M}\sum_{j=1}^{N} q_i q'_j \mathrm{Re}\left[\int_{z_0}^{z_i} dz\langle\partial_z\varphi(z,\bar{z})\left[\varphi(w_j,\bar{w}_j) - \varphi(u,\bar{u})\right]\rangle\right] =$$

$$\frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{N} q_i q'_j\left[\int_{z_0}^{z_i} dz\langle\partial_z\varphi(z,\bar{z})\left[\varphi(w_j,\bar{w}_j) - \varphi(u,\bar{u})\right]\rangle + cc\right] \tag{3.4}$$

Moreover, once $G_z(z; u, w)$ is known, one obtains also the correlator

$$G_{zu}(z; u) = \langle\partial_z\varphi(z,\bar{z})\partial_u\varphi(u,\bar{u})\rangle \tag{3.5}$$

as can be seen by differentiating both sides of equation (3.3) with respect to $u$. For these reasons, we will limit ourselves to the computation of $G_z(z; u, w)$.

To simplify the notations, we introduce the composite indices $I, J, K, \dots$. with $I = (i, s_i)$, $J = (j, s_j)$ etc. For instance, in this notation $\Omega_I(z)dz = \Omega_{(i,s_i)}(z)dz$. It is now possible to construct $G_z(z; u, w)$ starting from any third kind differential with two poles at $z = u, w$ and adding to it a suitable linear combination of holomorphic differentials. Using for example the third kind differential defined in eq. (2.17), we have in general:

$$G_z(z; u, w)dz = \nu_{uw}(z)dz + \sum_I B_I(u, w)\Omega_I(z)dz \tag{3.6}$$

Following the strategy of [38], we try to determine the coefficients $B_I(u,w)$ exploiting the fact that the fields $\varphi(z,\bar{z})$ are single-valued functions on $\Sigma_g$, so that they must obey the relations:

$$\oint_{\gamma_s} d\varphi = \oint_{\gamma_s} \partial_z \varphi dz + \oint_{\bar{\gamma}_s} \bar{\partial}_{\bar{z}} \varphi d\bar{z} = 0 \tag{3.7}$$

where the $\gamma_s$, $s = 1, \ldots, 2g$, form a basis of homology cycles on $\Sigma_g$. For consistency, the right hand side of eq. (3.4) has to be single-valued along the homology cycles. This condition is satisfied if and only if the correlator $G_z(z;u,w)$ is a differential of the third kind normalized in such a way that all its periods are purely imaginary. Let us denote this normalized differential with the symbol $\omega_{uw}(z)dz$. Thus we require that:

$$G_z(z;u,w)dz = \omega_{uw}(z)dz \tag{3.8}$$

The above equation determines $G_z(z;u,w)dz$ uniquely. Comparing in fact eq. (3.6) with the above equation and imposing the condition that $\nu_{uw}(z)dz + \sum_I B_I(u,w)\Omega_I(z)dz$ has imaginary periods around all homology cycles, one obtains the following linear system of $2g$ equations in the $2g$ real unknowns $\mathrm{Re}[B_I(u,w)]$ and $\mathrm{Im}[B_I(u,w)]$:

$$\mathrm{Re}\left[\oint_{\gamma_s} \nu_{uw}(z)dz + \sum_I B_I(u,w) \oint_{\gamma_s} \Omega_I(z)dz\right] = 0 \qquad\qquad s = 1, \ldots, 2g \tag{3.9}$$

Unfortunately, the construction of a basis of homology cycles on algebraic curves of the kind discussed here is a complicated mathematical problem. As a consequence, the periods $\oint_{\gamma_s} \nu_{uw}(z)dz$ and $\oint_{\gamma_s} \Omega_I(z)dz$ cannot be evaluated in a closed form. An exception is provided by the $Z_n$ symmetric curves, where these periods are expressed as definite integrals having the branch points as extrema. In the present case, however, even the positions of the branch points is unknown. The reason is that they are the roots of the polynomial $r(z) = 0$, where $r(z)$ is the resultant of the system (2.5) as discussed in the previous Section. Clearly, those roots cannot be computed analytically apart from simple cases.

To avoid these difficulties, we evaluate the coefficients $B_I(u,w)$ using the Riemann's bilinear equations [13] :

$$0 = \int_{\Sigma_g} \omega_{uw} \wedge \bar{\Omega}_J = \int_{\Sigma_g} d^2 z \omega_{uw}(z) \bar{\Omega}_I(\bar{z}) \tag{3.10}$$

where the $\bar{\Omega}_J(\bar{z})$'s denote the antiholomorphic differentials on $\Sigma_g$. In Appendix A we will show that the above relation holds if and only if $\omega_{uw}(z)dz$ is a normalized differential of the third kind with imaginary periods along the nontrivial homology cycles of $\Sigma_g$.

The integral in (3.10) is to be understood as a sum over all possible values of $z$ and $y$ for which the relation $F(z, y) = 0$ is satisfied. Supposing that the curve $\Sigma_g$ is nondegenerate, eq. (3.10) can be put in a form which is more convenient for explicit computations:

$$I = \int_{\mathbf{CP}^1} d^2 z \sum_{\alpha=0}^{n-1} f_{z\bar{z}}(z, \bar{z}; y^{(\alpha)}(z), \overline{y^{(\alpha)}(z)}) \tag{3.11}$$

A detailed proof of eq. (3.11) is provided in Appendix B (see also [14]). Once the polynomial $F(z, y)$ is given, it is possible to compute the integral (3.11) numerically. As a matter of fact, to evaluate the integrand

$$I_{z\bar{z}}(z, \bar{z}) = \sum_{\alpha=0}^{n-1} f_{z\bar{z}}(z, \bar{z}; y^{(\alpha)}(z), \overline{y^{(\alpha)}(z)}) \tag{3.12}$$

appearing in (3.11) at any given point $z \in \mathbf{CP}^1$, we just need to invert the equation $F(z, y) = 0$. The latter is a polynomial equations of degree $n$ from which one derives the $n$ roots $y^{(0)}(z), \ldots, y^{(n-1)}(z)$. The values of $\overline{y^{(0)}(z)}, \ldots, \overline{y^{(n-1)}(z)}$ are obtained by complex conjugation. The advantage of this strategy is that all branches of $y(z)$ enter symmetrically in the sum of eq. (3.12). Thus it is not necessary to know how these branches are exchanged at the branch points or any other information which requires the analytic continuation of $y(z)$.

Using the identity (3.8), we substitute in eq. (3.10) the expression of $G_z(z; u, w)dz$ in terms of the not normalized differential $\nu_{uw}(z)dz$ given by eq. (3.6). The solution of the obtained in this way linear equations for coefficients $B_I(u, w)$:

$$0 = \int_{\Sigma_g} \nu_{uw} \wedge \bar{\Omega}_J + \sum_I B_I(u, w) \int_{\Sigma_g} \Omega_I(z) \wedge \bar{\Omega}_J \tag{3.13}$$

is in the matrix form given by

$$B_I(u, w) = -\sum_J \left[ \int_{\Sigma_g} \nu_{uw} \wedge \bar{\Omega}_J \right] \left[ \int_{\Sigma_g} \Omega_J(z) \wedge \bar{\Omega}_I \right]^{-1} \tag{3.14}$$

In compact form the final result can be written up as ($\alpha, \beta, \gamma$ label sheets of the Riemann surface):

$$G_z^{(\alpha\beta\gamma)}(z; u, w) = \frac{1}{\det A_{IJ}} \det \left( \begin{array}{c|c} \dfrac{\nu_{uw}^{(\alpha\beta\gamma)}(z)}{\Phi_J^{(\alpha)}(u) - \Phi_J^{(\beta)}(w)} & \dfrac{\Omega_I^{(\alpha)}(z)}{A_{IJ}} \end{array} \right) \tag{3.15}$$

where

$$A_{IJ} \equiv A_{(i,s_i)(j,s_j)} = \int_{\mathbf{CP}^1} d^2z \sum_{\alpha=0}^{n-1} \frac{\left[y^{(\alpha)}(z)\right]^i \overline{\left[y^{(\alpha)}(z)\right]}^j}{\left|F_y(z, y^{(\alpha)}(z))\right|^2} z^{s_i} \bar{z}^{s_j} \tag{3.16}$$

and

$$\Phi_J^{(\beta)}(u) \equiv \Phi_{(j,s_j)}^{(\beta)}(u) = \sum_{k=0}^{n-1} \phi_{k,0}^{(\beta)}(u) \sum_{\alpha=0}^{n-1} \int_{\mathbf{CP}^1} d^2z \frac{\left[y^{(\alpha)}(z)\right]^{n-1-k} \overline{\left[y^{(\alpha)}(z)\right]}^j \bar{z}^{s_j}}{\left|F_y(z, y^{(\alpha)}(z))\right|^2 (z-u)} \tag{3.17}$$

In eq. (3.16) the range of the integers $i, j, s_i$ and $s_j$ is given by eq. (2.22). Moreover, we notice that the integrals in eqs. (3.16) and (3.17) are convergent, since the integrands have at most harmless poles of the first order. These singularities occur only in the integrals (3.17) at the points $z = u, w$ and at infinity. The singularity at infinity is present only when $k = 0$ and $p > 1$. Clearly, the discussion following eq. (3.11) applies also to the particular case of the integrals in (3.16) and (3.17), which can thus be computed numerically without problems of analytical continuation.

Eqs. (3.15)-(3.17) provide an explicit representation of the correlator $G_z(z; u, w)$, where neither the knowledge of a basis of homology cycles nor the analytic continuation of multivalued functions are necessary. This is a great advantage, since the latter are formidable problems in the case of general curves like those discussed in this paper. Of course, the integrals (3.16)-(3.17) are complicated and cannot be computed analytically, but only numerically, exploiting for instance the recipe explained after eq. (3.11).

To conclude this Section, we study the behavior of $G_z(z; u, w)dz$ for large values of $u$, showing that there are no spurious divergences in this case. From eqs. (2.20) and (3.17) one realizes that $G_z(z; u, w)dz$ depends on $u$ through the function

$$f(u) = \frac{\phi_{k,0}(u)}{z-u} \tag{3.18}$$

When $u$ becomes large, one can write:

$$f(u) = f^{\text{div}}(u) + f^{\text{fin}}(u) \tag{3.19}$$

where $f^{\text{div}}(u)$ diverges in $u = \infty$ while $f^{\text{fin}}(u)$ remains finite. It is easy to see that:

$$f^{\text{div}}(u) = -\frac{\phi_{k,0}(u)}{u} \sum_{i=0}^{pk-2} \left(\frac{z}{u}\right)^i \tag{3.20}$$

with

$$k > 0 \qquad \text{if} \qquad p > 1 \tag{3.21}$$

and

$$k > 1 \qquad \text{if} \qquad p = 1 \tag{3.22}$$

Using the above formula, we extract the diverging contributions $G_z^{\rm div}(z;u,w)dz$ which are present in the correlator $G_z(z;u,w)dz$. To this purpose, it is convenient to define the composite index:

$$\overline{I}(k,i) = (n-1-k,i) \tag{3.23}$$

Hereafter, the dependence of $\overline{I}$ on $k$ and $i$ will be only understood for simplicity. After some calculations one finds:

$$G_z^{\rm div}(z;u,w) = \frac{1}{\det A_{IJ}} \sum_{k=1+\delta_{p,1}}^{n-1} \sum_{i=0}^{pk-2} \frac{\phi_{k,0}(u)}{u} \det \left( \begin{array}{c|c} \frac{\Omega_{\overline{I}}(z)}{A_{\overline{I}J}} & \frac{\Omega_K(z)}{A_{JK}} \end{array} \right) \tag{3.24}$$

Clearly

$$\det \left( \begin{array}{c|c} \frac{\Omega_{\overline{I}}(z)}{A_{\overline{I}J}} & \frac{\Omega_K(z)}{A_{JK}} \end{array} \right) = 0 \tag{3.25}$$

showing that $G_z(z;u,w)dz$ has no singularities for large values of $u$ as desired. The absence of divergences when $w$ approaches infinity can be proved in the same way.

## 4. EXAMPLES

A first non-trivial example of curves which can be explicitly worked out is provided by the algebraic curve:

$$y^3 + 3py - 2q = 0 \tag{4.1}$$

$q(z)$ is a polynomial of degree $n_0 = 6$, while $p(z)$ has degree $n_1 \leq \frac{2n_0}{3} = 4$. Thus, exploiting the formulas of Section 2, one realizes that the curve (4.1) corresponds to the particular case $p = 2$ and $g = 4$. For future convenience let us put:

$$\xi_\pm(z) = \sqrt[3]{q \pm \sqrt{q^2 + p^3}} \tag{4.2}$$

Solving equation (4.1) with respect to $y$ one finds:

$$y^\alpha = \epsilon^\alpha \xi_+ + \epsilon^{2\alpha} \xi_- \qquad \alpha = 0,1,2 \qquad \epsilon = e^{\frac{2\pi i}{3}} \tag{4.3}$$

The four holomorphic differentials of the above curve are:

$$\Omega_{(0,s_0)}(z)dz = \frac{z^{s_0}dz}{F_y} \qquad\qquad \Omega_{(1,1)}(z)dz = \frac{ydz}{F_y} \tag{4.4}$$

where $s_0 = 0, 1, 2$ and $F_y = 3(y^2 + p)$. The differential of the third kind of eq. (2.17) becomes in this particular case:

$$\nu_{uw}(z)dz = \frac{y^2(z) + y(z)y(u) + y^2(u) + 3p(u)}{F_y(z, y(z))(z-u)} - (u \leftrightarrow w) \tag{4.5}$$

At this point, we are ready to compute the matrix elements $A_{IJ}$ and $\Phi_I$. They will be expressed as integrals of single-valued forms on $\mathbf{CP}^1$. To simplify the notations, let us first define the following functions:

$$\mathcal{F}(z) = \prod_{\alpha=0}^{2} F_y(z, y^{(\alpha)}(z)) \tag{4.6}$$

$$a(z,\bar{z}) = \xi_+^2(z)\overline{\xi_+^2(z)} + \xi_-^2(z)\overline{\xi_-^2(z)} + p(z)\overline{p(z)} \tag{4.7}$$

$$b(z,\bar{z}) = \xi_+^2(z)\overline{\xi_-^2(z)} - p(z)\overline{\xi_+^2(z)} - \overline{p(z)}\xi_-^2(z) \tag{4.8}$$

$$c(z,\bar{z}) = \overline{\xi_+^2(z)}\xi_-^2(z) - p(z)\overline{\xi_-^2(z)} - \overline{p(z)}\xi_+^2(z) \tag{4.9}$$

$$A(z,\bar{z}) = 3(a^2 - bc) \tag{4.10}$$

$$B(z,\bar{z}) = 3\left[\xi_+\left(b^2 - ac\right) + \xi_-\left(c^2 - ab\right)\right] \tag{4.11}$$

$$C(z,\bar{z}) = \left[\overline{\xi_+}\left(c^2 - ab\right) + \overline{\xi_-}\left(b^2 - ac\right)\right] \tag{4.12}$$

$$D(z,\bar{z}) = 3\left[\left(\xi_+\overline{\xi_+} + \xi_-\overline{\xi_-}\right)(a^2 - bc) + \xi_-\overline{\xi_+}(b^2 - ac) + \xi_+\overline{\xi_-}(c^2 - ab)\right] \tag{4.13}$$

$$E(z,\bar{z}) = 3\left[2\xi_+\xi_-(a^2 - bc) + \xi_-^2(b^2 - ac) + \xi_+^2(c^2 - ab)\right] \tag{4.14}$$

$$F(z,\bar{z}) =$$

$$3\left[\left(\xi_-^2\overline{\xi_+} + \xi_+^2\overline{\xi_-}\right)(a^2 - bc) + \left(\xi_+^2\overline{\xi_+} - 2p\overline{\xi_-}\right)(b^2 - ac) + \left(\xi_-^2\overline{\xi_-} - 2p\overline{\xi_+}\right)(c^2 - ab)\right] \tag{4.15}$$

The above functions are not meromorphic, since they depend both on $z$ and $\bar{z}$, but are single-valued on $\mathbf{CP}_1$. To prove this fact, we notice that under the most general monodromy transformation for going from one branch of the curve to another $\xi_+$ behaves as

$$\xi_+(z) \to \epsilon^\alpha \xi_\pm(z) \qquad\qquad \alpha = 0, 1, 2 \mod 3 \tag{4.16}$$

as can be easily seen from the explicit expression of $\xi_+$ given by eq. (4.2). The above equation fixes also the monodromy properties of $\xi_-(z)$. In fact, since $\xi_+(z)$ and $\xi_-(z)$ must satisfy the relation $\xi_+(z)\xi_-(z) = -p(z)$, where $p(z)$ is a singlevalued function of $z$, it is clear from eq. (4.16) that $\xi_-(z)$ should transform as follows:

$$\xi_-(z) \to \epsilon^{2\alpha}\xi_\mp(z) \tag{4.17}$$

Moreover, taking the complex conjugate of both members of eqs. (4.16) and (4.17), one finds:

$$\overline{\xi_+(z)} \to \epsilon^{2\alpha}\overline{\xi_\pm(z)} \tag{4.18}$$

$$\overline{\xi_-(z)} \to \epsilon^{\alpha}\overline{\xi_\mp(z)} \tag{4.19}$$

It is now easy to check that the functions defined in eqs. (4.10)–(4.15) are invariant under the transformations (4.16)–(4.19), so that they are single-valued on $\mathbf{CP}^1$. In terms of these functions, the matrix elements $A_{IJ}$ and $\Phi_I$ appearing in the normalized differential of the third kind (3.15) read as follows:

$$A_{(0,s_0)(0,s_0')} = \int_{\mathbf{CP}^1} d^2z \frac{z^{s_0}\,\bar{z}^{s_0'}}{|\mathcal{F}(z)|^2} A(z,\bar{z}) \tag{4.20}$$

$$A_{(1,0)(0,s_0)} = \int_{\mathbf{CP}^1} d^2z \frac{\bar{z}^{s_0}}{|\mathcal{F}(z)|^2} B(z,\bar{z}) \tag{4.21}$$

$$A_{(0,s_0)(1,0)} = \int_{\mathbf{CP}^1} d^2z \frac{z^{s_0}}{|\mathcal{F}(z)|^2} C(z,\bar{z}) \tag{4.22}$$

$$A_{(1,0)(1,0)} = \int_{\mathbf{CP}^1} d^2z \frac{D(z,\bar{z})}{|\mathcal{F}(z)|^2} \tag{4.23}$$

$$\Phi_{(0,s_0)}(u) = \int_{\mathbf{CP}^1} d^2z \frac{\bar{z}^{s_0}E(z,\bar{z})}{|\mathcal{F}(z)|^2\,(z-u)} +$$

$$y(u)\int_{\mathbf{CP}^1} d^2z \frac{\bar{z}^{s_0}B(z,\bar{z})}{|\mathcal{F}(z)|^2\,(z-u)} + (y^2(u)+3p(u))\int_{\mathbf{CP}^1} d^2z \frac{\bar{z}^{s_0}A(z,\bar{z})}{|\mathcal{F}(z)|^2\,(z-u)} \tag{4.24}$$

$$\Phi_{(1,0)}(u) = \int_{\mathbf{CP}^1} d^2z \frac{F(z,\bar{z})}{|\mathcal{F}(z)|^2\,(z-u)} +$$

$$y(u)\int_{\mathbf{CP}^1} d^2z \frac{D(z,\bar{z})}{|\mathcal{F}(z)|^2\,(z-u)} + (y^2(u)+3p(u))\int_{\mathbf{CP}^1} d^2z \frac{C(z,\bar{z})}{|\mathcal{F}(z)|^2\,(z-u)} \tag{4.25}$$

Substituting the matrix elements (4.20)–(4.25) and the third kind differential (4.5) in eq. (3.15), we obtain the explicit expression of the correlator (3.3). As we see, all the integrands appearing in the right hand sides of eqs. (4.20)–(4.25) are one-valued on the complex sphere as desired, since they depend on the one-valued functions (4.6)–(4.15). In this way the integrals in (4.20)–(4.25) can be numerically evaluated without the problem of performing the analytic continuation of $y(z)$. Also the problem of constructing a basis of homology cycles has been avoided.

Let us now briefly discuss the simple case of the $Z_n$ algebraic curves of the kind:

$$y^n = p_0(z) \tag{4.26}$$

where $p_0(z)$ is a polynomial of degree $n_0 = n, 2n, 3n, \ldots$. The genus of these curves is provided by the general formula (2.23). For $p > 1$, i.e. $n_0 > n$, the zero modes are:

$$\Omega_{j,s_j}(z)dz = \frac{z^{s_j} y^j(z)}{y^{n-1}} dz \tag{4.27}$$

The matrix elements $A_{IJ}$ are given by:

$$A_{(i,s_i)(j,s_j)} = 0 \qquad\qquad i \neq j \tag{4.28}$$

$$A_{(j,s_j)(j,s_j)} = n \int_{\mathbf{CP}^1} d^2 z z^{s_j} \bar{z}^{s_j} |y\bar{y}|^{1-n+j} \tag{4.29}$$

Clearly, the integrand in the right hand side of the above equation is single-valued. In the same way one can discuss the elements $\Phi_I(u)$. This is a straightforward exercise which will not be performed here. Instead, we compute on a $Z_n$ symmetric curve the propagator

$$G(z; a) = \langle \varphi(z, \bar{z}) \varphi(a, \bar{a}) \rangle \tag{4.30}$$

where $a$ is one of the $pn$ branch points of the curve (4.26). The starting point is the Weierstrass kernel at a branch point, which in the case of a $Z_n$ symmetric curve takes a particularly simple form (put $y(a) = 0$ in (2.16) ):

$$K_1(z, a)dz = \frac{1}{n} \frac{dz}{z - a} \tag{4.31}$$

We define the following function on the complex sphere:

$$\mathcal{G}(z, w) = \frac{1}{2n} \log \left( \frac{|z - w|^2}{(1 + z\bar{z})} \right) \tag{4.32}$$

which has only logarithmic singularities for $z \to \infty$. $\mathcal{G}(z, w)$ satisfies the identity:

$$\partial_z \partial_{\bar{z}} \mathcal{G}(z, w) = \frac{\pi}{n} \delta^{(2)}_{z\bar{z}}(z - w) - \frac{1}{2n} \frac{1}{(1 + z\bar{z})^2} \tag{4.33}$$

where $\delta^{(2)}_{z\bar{z}}(z-w)$ is defined by

$$\int_{\mathbf{CP}^1} d^2z f(z,\bar{z})\delta^{(2)}_{z\bar{z}}(z-w) = f(w,\bar{w}). \tag{4.34}$$

(4.33) is the relation satisfied by a Green function on the complex sphere with the metric $ds^2 = \frac{dzd\bar{z}}{(1+z\bar{z})^2}$. Let us now consider a two-form $f(z,\bar{z}) = f_{z\bar{z}}(z,\bar{z})\frac{d\bar{z}\wedge z}{2i}$ on $\Sigma_g$ with sufficiently smooth behavior. $f_{z\bar{z}}$ is in general a multivalued function on $\mathbf{CP}_1$. Exploiting eq. (3.11), we have:

$$\int_{\Sigma_g} d^2w f_{w\bar{w}}(w,\bar{w};y,\bar{y})\tilde{\delta}^{(2)}_{z\bar{z}}(z-w) \equiv \sum_{\alpha=0}^{n-1}\int_{\mathbf{CP}_1} d^2w f_{w\bar{w}}(w,\bar{w};y^{(\alpha)}(w),\overline{y^{(\alpha)}(w)})\delta^{(2)}_{z\bar{z}}(z-w) =$$

$$= \sum_{\alpha=0}^{n-1} f^{(\alpha)}_{z\bar{z}}(z,\bar{z}). \tag{4.35}$$

On $Z_n$ symmetric curves near branch points all the branches of $y(z)$ coincide. For this reason, if $z \to a$ eq. (4.35) becomes:

$$\int_{\Sigma_g} d^2w f_{w\bar{w}}(w,\bar{w};y,\bar{y})\tilde{\delta}^{(2)}_{z\bar{z}}(a-w) = n\ f_{z\bar{z}}(z,\bar{z})\big|_{\substack{z=a\\ \bar{z}=\bar{a}}} \tag{4.36}$$

Thus, the proper definition of a $\delta-$function on $\Sigma_g$ at a branch point is $\frac{1}{n}\tilde{\delta}^{(2)}_{z\bar{z}}(z-a)$.

We try now an ansatz

$$G(z;a) = \mathcal{G}(z,a) - \frac{1}{A}\int_{\Sigma_g} d^2w\tilde{\rho}_{w\bar{w}}\mathcal{G}(z,w) \tag{4.37}$$

for the propagator (4.30). In the obvious way $\mathcal{G}$ is treated in (4.37) as an object on $\Sigma_g$ The metric $\tilde{\rho}_{w\bar{w}}$ is given by eq. (2.25), while

$$A = \int_{\Sigma_g} d^2w\tilde{\rho}_{w\bar{w}} \tag{4.38}$$

is the area of $\Sigma_g$. We notice that, by construction, the metric $\tilde{\rho}_{w\bar{w}}$ is $Z_n$ symmetric, so that all its branches coincide. As a consequence:

$$\tilde{\rho}_{w\bar{w}}(w,\bar{w};y^{(\alpha)}(w)\overline{y^{(\alpha)}(w)}) = \rho_{w\bar{w}}(w,\bar{w}) \tag{4.39}$$

where $\rho_{w\bar{w}}(w,\bar{w})$ is a tensor on $\mathbf{CP}_1$. Thus, exploiting again eq. (3.11), we can put:

$$A = n\int_{\mathbf{CP}_1} d^2w\rho_{w\bar{w}}(w,\bar{w}) \tag{4.40}$$

Applying now the Laplacian on $\Sigma_g$ to both members of eq. (4.37) and using eq. (4.33), we find that $G(z,a)$ satisfies the following identity:

$$\partial_z\partial_{\bar z}G(z,a) = \pi\tilde\delta^{(2)}_{z\bar z}(z-a) - \frac{1}{A}\tilde\rho_{z\bar z}(z,\bar z) \tag{4.41}$$

As explained above, the $\delta-$function on the algebraic curve near a branch point $a$ is exactly $\frac{1}{n}\delta^{(2)}_{z\bar z}(z-a)$. Moreover, all the branches of the metric $\tilde\rho_{z\bar z}$ coincide with $\rho_{z\bar z}(z,\bar z)$. Thus we conclude that eq. (4.37) is the desired propagator of the scalar fields when one of the fields is located at a branch point.

## 5. CONCLUSIONS

In this paper the correlator $G_z(z;u,w)dz$ defined in eq. (3.3) has been computed on the general algebraic curves (2.2). This Green function plays a fundamental role in the theory of massless scalar fields, since all other correlation functions can be written in terms of its derivatives or integrals as shown by eqs. (3.4) and (3.5). The expression of $G_z(z;u,w)dz$ given in eq. (3.15) depends explicitly on the constant parameters $A_{s,m}$ of the polynomial (2.2) as desired. In fact, it contains the Weierstrass kernel and the holomorphic differentials which have been derived in eqs. (2.16) and (2.21) respectively. The calculation of the coefficients $A_{IJ}$ and $\phi^\alpha_J(w)$, in principle more tricky, has been reduced to the evaluation of the integrals (3.16) and (3.17) on the complex sphere. The latter may very complicated, especially if the polynomial (2.2) has high degree $n$, but they do not hide outstanding technical difficulties and can be performed at least numerically (see eq. (3.11) and the following discussion). Actually, the integrands appearing in eqs. (3.16) and (3.17) should also be single-valued, because they consists in the sum over all branches of multivalued complex forms. This fact has been explicitly shown in the particular example of an algebraic curve of genus four worked out in Section 4. However, this example shows also that the single-valuedness is realized in a very non-trivial way. Let us notice that the absence of technical problems in the computation of $G_z(z;u,w)dz$ was not granted a priori. For instance, choosing the set of equations (3.9) in order to determine this Green function, we would have not been able to compute the necessary line integrals along the homology cycles even numerically, since it is not known how to construct an homology basis on general algebraic curves.

Concluding, the formula (3.15), which gives the explicit form of the canonical third kind differential with purely imaginary periods, is new in the theory of algebraic curves and has many potential applications beyond the theory of free scalar fields. Let us remember in fact that this canonical differential is well known in the mathematical literature and has already been widely used in physical applications of Riemann surfaces. Also the propagator at the branch points of eq. (4.37) may have some interest in string theories. Finally, we have developed in this paper several techniques which might be useful for whoever is wishing to work on general algebraic curves.

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**Appendix A.**

In this Appendix we show that the condition (3.13) is valid if and only if $\omega_{uw}(z)dz$ is a differential of the third kind normalized in such a way that all its periods around the homology cycles are imaginary. In the proof, it will be convenient to use the representation of the Riemann surface $\Sigma_g$ as a polygon $M = \prod_{i=1}^{g} a_j b_j a_j^{-1} b_j^{-1}$, where $\{a_i, b_i | i = 1, \ldots, g\}$ is a canonical basis of homology cycles [12]. Accordingly, the notation will be changed with respect to the rest of the paper. Let $x, y$ be real coordinates on $M$ and $z = x + iy$, $\bar{z} = x - iy$. The basis of canonical holomorphic differentials on $\Sigma_g$ will be denoted as follows:

$$\omega_i = \omega_i(z)dz \qquad\qquad i = 1, \ldots, g \tag{A.1}$$

The $\omega_i$ are normalized in such a way that:

$$\oint_{a_i} \omega_j = \delta_{ij} \qquad\qquad \oint_{b_i} \omega_j = T_{ij} \tag{A.2}$$

for $i, j = 1, \ldots, g$. The $T_{ij} = T_{ji}$ are elements of the period matrix $T$. This is a symmetric $g \times g$ matrix with positive definite imaginary part. Since $M$ is simply connected, there exist holomorphic functions $f_j(z)$ such that

$$df_j(z) = \omega_j(z)dz \tag{A.3}$$

In the above equation $d$ denotes the external derivative operator such that $d^2 = 0$. Analogously, one can define the antiholomorphic differentials $\bar{\omega}_j = \bar{\omega}_j(\bar{z})d\bar{z}$ and functions $\bar{f}_j(\bar{z})$.

Finally, third kind differentials will be denoted with the symbol $\omega_{PQ} = \omega_{PQ}(z)dz$, with $P, Q \in M$. We are now ready to prove the following

**Proposition:** The conditions

$$\int_M \omega_{PQ} \wedge \bar{\omega}_j = 0 \qquad\qquad j = 1, \ldots, g \tag{A.4}$$

are verified iff $\omega_{PQ}$ is normalized in such a way that all its periods are imaginary, i.e.

$$\overline{\oint_{a_i} \omega_{PQ}} = -\oint_{a_i} \omega_{PQ} \qquad\qquad \overline{\oint_{b_i} \omega_{PQ}} = -\oint_{b_i} \omega_{PQ} \tag{A.5}$$

**Proof:** We assume first that eq. (A.4) is true. Introducing the antiholomorphic functions $\bar{f}_j(\bar{z})$ and exploiting the Stoke's theorem after an integration by parts, we obtain:

$$0 = -\int_{\partial M} \bar{f}_j \omega_{PQ} + \int_M \bar{f}_j d\omega_{PQ} \tag{A.6}$$

where $\partial M$ is the boundary of the polygon $M$. Using the Riemann's bilinear formula [13]:

$$\int_{\partial M} \bar{f}_j \omega_{PQ} = \sum_{l=1}^{g} \left[ \overline{\oint_{a_l} \omega_j} \oint_{b_l} \omega_{PQ} - \overline{\oint_{b_l} \omega_j} \oint_{a_l} \omega_{PQ} \right] \tag{A.7}$$

and remembering eq. (A.2), the first term in the right side of eq. (A.6) becomes:

$$\int_{\partial M} \bar{f}_j \omega_{PQ} = \oint_{b_j} \omega_{PQ} - \sum_{l=1}^{g} \overline{T}_{lj} \oint_{a_l} \omega_{PQ} \tag{A.8}$$

The second term gives instead:

$$\int_M \bar{f}_j d\omega_{PQ} = \int_M \bar{f}_j (\bar{\partial}_{\bar{z}} \omega_{PQ}) d\bar{z} \wedge dz \tag{A.9}$$

where $\bar{\partial}_{\bar{z}}$ denotes partial derivative with respect to $\bar{z}$. At this point, we notice that:

$$\bar{\partial}_{\bar{z}} \omega_{PQ}(z) = \pi \left[ \delta^{(2)}(z, P) - \delta^{(2)}(z, Q) \right] \tag{A.10}$$

and $d\bar{z} \wedge dz = 2idx \wedge dy = 2id^2x$, where $\delta^{(2)}(z, w)$ is the Dirac $\delta-$ function on $M$ defined in such a way that

$$\int_M f(z, \bar{z}) \delta^{(2)}(z, w) d^2x = f(w, \bar{w}) \tag{A.11}$$

Thus:

$$\int_M \bar{f}_j d\omega_{PQ} = 2\pi i \left( \bar{f}_j(\bar{P}) - \bar{f}_j(\bar{Q}) \right) = 2\pi i \overline{\int_Q^P \omega_j} \tag{A.12}$$

Substituting eq. (A.8) and (A.12) in (A.6) we obtain the relation:

$$\oint_{b_j} \omega_{PQ} - \sum_{l=1}^{g} \overline{T}_{lj} \oint_{a_l} \omega_{PQ} = 2\pi i \overline{\int_Q^P \omega_j} \tag{A.13}$$

On the other side, exploiting similar arguments to evaluate the integral $\int \omega_{PQ} \wedge \omega_j$ (which necessarily vanishes since $\omega_{PQ}$ is a meromorphic differential and $dz \wedge dz = 0$) one has [13]:

$$\oint_{b_j} \omega_{PQ} - \sum_{l=1}^{g} T_{lj} \oint_{a_l} \omega_{PQ} = 2\pi i \int_Q^P \omega_j \tag{A.14}$$

Eqs. (A.13) and (A.14) provide a system of linear equations for the $2g$ unknown periods of $\omega_{PQ}$ around the homology cycles. After solving it, one finds:

$$\oint_{a_k} \omega_{PQ} = 2\pi i \sum_{j=1}^{g} (\mathrm{Im}[T])^{-1}_{jk} \, \mathrm{Im} \left[ \int_Q^P \omega_j \right] \tag{A.15}$$

$$\oint_{b_j} \omega_{PQ} = 2\pi i \sum_{l=1}^{g} \mathrm{Re}[T_{lj}] \sum_{k=1}^{g} \left\{ (\mathrm{Im}[T])^{-1}_{kl} \, \mathrm{Im} \left[ \int_Q^P \omega_k \right] \right\} + 2\pi i \mathrm{Re} \left[ \int_Q^P \omega_j \right] \tag{A.16}$$

where $\mathrm{Im}[T]$ is the imaginary part of the period matrix and $(\mathrm{Im}[T])^{-1}_{kl}$ are the elements of its inverse. From eqs. (A.15) and (A.16) it is clear that the periods of $\omega_{PQ}$ are purely imaginary as desired.

Conversely, let us assume that (A.5) is true. Evaluating as before $\int_M \omega_{PQ} \wedge \bar{\omega}_j$ we obtain:

$$\int_M \omega_{PQ} \wedge \bar{\omega}_j = \sum_{l=1}^{g} \overline{T}_{lj} \oint_{a_l} \omega_{PQ} - \oint_{b_j} \omega_{PQ} + 2\pi i \overline{\int_Q^P \omega_j} \tag{A.17}$$

Moreover, the complex conjugate of eq. (A.14) becomes with the help of eqs. (A.5):

$$\oint_{b_j} \omega_{PQ} = \sum_{l=1}^{g} \overline{T}_{jl} \oint_{a_l} \omega_{PQ} + 2\pi i \overline{\int_Q^P \omega_j} \tag{A.18}$$

Substituting the above value of the $b_j-$periods of $\omega_{PQ}$ in eq. (A.17) we recover exactly eq. (A.4).

**Appendix B.**

In this Appendix we show that the surface integral over $\Sigma_g$ of a tensor $f_{z\bar{z}}(z,y)$ can be rewritten in the form (3.11). The material presented here can be found in standard textbooks [14]– [39]. To keep the notations as simple as possible, we will omit to write explicitly the dependence of $f_{z\bar{z}}(z,y)$ on the conjugate variables $\bar{z},\bar{y}$. We start from

$$I = \int_{\Sigma_g} d^2 z f_{z\bar{z}}(z,y) \tag{B.1}$$

supposing that $f_{z\bar{z}}(z,y)$ is such that the above integral is convergent. Integrals of the kind (B.1) are to be understood as a sum over all values of $z \in \mathbf{CP}^1$ and $y$ for which eq. (2.1) is satisfied, i. e.:

$$I = \int_{\substack{z\in\mathbf{CP}^1\\ F(z,y)=0}} f_{z\bar{z}}(z,y) \tag{B.2}$$

Since we assume that our curves are not degenerate, it is possible to write:

$$I = \int_{\mathbf{CP}^1} d^2 z \int_{\mathbf{CP}^1} d^2 y \sqrt{G} f_{z\bar{z}}(z,y) \delta^{(2)}(F(z,y)) \tag{B.3}$$

where $\sqrt{G}$ is the determinant of the metric in the $y$ domain and the Dirac $\delta-$function is defined as follows:

$$\int_{\mathbf{CP}^1} d^2 y \sqrt{G} f(y) \delta^{(2)}(y-y') = f(y') \tag{B.4}$$

It is now convenient to choose a conformally flat metric

$$G_{yy} = G_{\bar{y}\bar{y}} = 0 \qquad\qquad G_{y\bar{y}} = G_{\bar{y}y} \tag{B.5}$$

so that eq. (B.3) becomes:

$$I = \int_{\mathbf{CP}^1} d^2 z \int_{\mathbf{CP}^1} d^2 y f_{z\bar{z}}(z,y) \delta^{(2)}_{y\bar{y}}(F(z,y)) \tag{B.6}$$

Locally, we have that:

$$\delta^{(2)}_{y\bar{y}}(F(z,y)) = \frac{1}{4}\partial_y \partial_{\bar{y}} \log|F(z,y)|^2 \tag{B.7}$$

Exploiting the relation

$$F(z,y) = \prod_{\alpha=0}^{n-1}(y - y^{(\alpha)}(z)) \tag{B.8}$$

we see that the zeros of $F(z,y)$ are concentrated at the branches of $y(z)$, where $F(z,y)$ can be approximated as follows:

$$F(z,y) \sim F_y(z, y^{(\alpha)}(z))(y - y^{(\alpha)}(z)) + \dots \tag{B.9}$$

As a consequence:

$$I = \sum_{\alpha=0}^{n-1} \int_{\mathbf{CP}^1} d^2z \int_{\mathbf{CP}^1} d^2y f_{z\bar{z}}(z, y^{(\alpha)}(z)) \delta^{(2)}_{y\bar{y}} \left( F_y(z, y^{(\alpha)}(z))(y - y^{(\alpha)}(z)) \right) \tag{B.10}$$

Performing now the change of variables

$$F_y(z, y^{(\alpha)}(z))(y - y^{(\alpha)}(z)) = y' \tag{B.11}$$

and integrating over $y'$, we obtain the desired result:

$$I = \int_{\mathbf{CP}^1} \sum_{\alpha=0}^{n-1} f_{z\bar{z}}(z, y^{(\alpha)}(z)) \tag{B.12}$$

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